

Does there exist an algorithm which to each Diophantine equation assigns an integer which is greater than the number (heights) of integer solutions, if these solutions form a finite set?

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**Abstract.** Let  $E_n = \{x_i = 1, x_i + x_j = x_k, x_i \cdot x_j = x_k : i, j, k \in \{1, \dots, n\}\}$ . If Matiyasevich's conjecture on finite-fold Diophantine representations is true, then for every computable function  $f : \mathbb{N} \rightarrow \mathbb{N}$  there is a positive integer  $m(f)$  such that for each integer  $n \geq m(f)$  there exists a system  $S \subseteq E_n$  which has at least  $f(n)$  and at most finitely many solutions in integers  $x_1, \dots, x_n$ . This conclusion contradicts to the author's conjecture on integer arithmetic, which implies that the heights of integer solutions to a Diophantine equation are computably bounded, if these solutions form a finite set.

**Key words and phrases:** computable function, Davis-Putnam-Robinson-Matiyasevich theorem, finite-fold Diophantine representation, Matiyasevich's conjecture, system of Diophantine equations.

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## 1. Introduction

The heights of integer solutions to a Diophantine equation

$$ax^2 + bxy + cy^2 + dx + ey + f = 0 \quad (1)$$

are bounded from above by  $20 \cdot (\max(|a|, |b|, |c|, |d|, |e|, |f|))^4$ , if equation (1) has at most finitely many integer solutions, see [10], [8, p. 17, Theorem 1.14], and [9]. Every Diophantine equation of degree at most  $n$  has the form

$$\sum_{\substack{i_1, \dots, i_k \in \{0, \dots, n\} \\ i_1 + \dots + i_k \leq n}} a(i_1, \dots, i_k) \cdot x_1^{i_1} \cdot \dots \cdot x_k^{i_k} = 0 \quad (2)$$

where  $a(i_1, \dots, i_k)$  denote integers.

**Observation 1.** *For each positive integer  $b$ , there are at most finitely many equations (2) which satisfy*

$$\max\left(\{k, n\} \cup \left\{|a(i_1, \dots, i_k)| : (i_1, \dots, i_k \in \{0, \dots, n\}) \wedge (i_1 + \dots + i_k \leq n)\right\}\right) \leq b$$

We state the following double problem: *Does there exist a computable function of*

$$\max\left(\{k, n\} \cup \left\{|a(i_1, \dots, i_k)| : (i_1, \dots, i_k \in \{0, \dots, n\}) \wedge (i_1 + \dots + i_k \leq n)\right\}\right)$$

*which bounds the number (heights) of integer solution to equation (2), if these solutions form a finite set?*

The existence of such bounds is discussed in this article. By Observation 1, the stated problem is equivalent to the double problem from the title of the article.

## 2. Small systems of Diophantine equations with a large number of integer solutions

Let  $E_n = \{x_i = 1, x_i + x_j = x_k, x_i \cdot x_j = x_k : i, j, k \in \{1, \dots, n\}\}$ . The following system

$$\begin{cases} x_1 \cdot x_1 & = & x_1 \\ & \dots & \\ x_n \cdot x_n & = & x_n \end{cases}$$

has exactly  $2^n$  solutions in integers  $x_1, \dots, x_n$ . If  $n \geq 10$ , then  $1156 \cdot 2^{n-10} > 2^n$  and there is a simply defined system  $S \subseteq E_n$  which has exactly  $1156 \cdot 2^{n-10}$  solutions in integers  $x_1, \dots, x_n$ , see [1]. We strengthen this results assuming an old conjecture due to Yu. Matiyasevich.

The Davis-Putnam-Robinson-Matiyasevich theorem states that every recursively enumerable set  $\mathcal{M} \subseteq \mathbb{N}^n$  has a Diophantine representation, that is

$$(a_1, \dots, a_n) \in \mathcal{M} \iff \exists x_1, \dots, x_m \in \mathbb{N} \ W(a_1, \dots, a_n, x_1, \dots, x_m) = 0 \quad (\text{R})$$

for some polynomial  $W$  with integer coefficients, see [5] and [4]. The polynomial  $W$  can be computed, if we know a Turing machine  $M$  such that, for all  $(a_1, \dots, a_n) \in \mathbb{N}^n$ ,  $M$  halts on  $(a_1, \dots, a_n)$  if and only if  $(a_1, \dots, a_n) \in \mathcal{M}$ , see [5] and [4].

The representation (R) is said to be finite-fold if for any  $a_1, \dots, a_n \in \mathbb{N}$  the equation  $W(a_1, \dots, a_n, x_1, \dots, x_m) = 0$  has at most finitely many solutions  $(x_1, \dots, x_m) \in \mathbb{N}^m$ . Yu. Matiyasevich conjectures that each recursively enumerable set  $\mathcal{M} \subseteq \mathbb{N}^n$  has a finite-fold Diophantine representation, see [3, pp. 341–342], [6, p. 42] and [7, p. 79].

Before the main Theorem 1, we need an algebraic lemma together with introductory matter.

Let  $D(x_1, \dots, x_p) \in \mathbb{Z}[x_1, \dots, x_p]$ . For the Diophantine equation  $2 \cdot D(x_1, \dots, x_p) = 0$ , let  $M$  denote the maximum of the absolute values of its coefficients. Let  $\mathcal{T}$  denote the family of all polynomials  $W(x_1, \dots, x_p) \in \mathbb{Z}[x_1, \dots, x_p]$  whose all coefficients belong to the interval  $[-M, M]$  and  $\deg(W, x_i) \leq d_i = \deg(D, x_i)$  for each  $i \in \{1, \dots, p\}$ . Here we consider the degrees of  $W(x_1, \dots, x_p)$  and  $D(x_1, \dots, x_p)$  with respect to the variable  $x_i$ .

We choose any bijection  $\tau : \{p+1, \dots, \text{card}(\mathcal{T})\} \longrightarrow \mathcal{T} \setminus \{x_1, \dots, x_p\}$ . Let  $\mathcal{H}$  denote the family of all equations of the forms

$$x_i = 1, x_i + x_j = x_k, x_i \cdot x_j = x_k \quad (i, j, k \in \{1, \dots, \text{card}(\mathcal{T})\})$$

which are polynomial identities in  $\mathbb{Z}[x_1, \dots, x_p]$  if

$$\forall s \in \{p+1, \dots, \text{card}(\mathcal{T})\} \quad x_s = \tau(s)$$

There is a unique  $q \in \{p+1, \dots, \text{card}(\mathcal{T})\}$  such that  $\tau(q) = 2 \cdot D(x_1, \dots, x_p)$ . For each ring  $\mathbf{K}$  extending  $\mathbb{Z}$  the system  $\mathcal{H}$  implies  $2 \cdot D(x_1, \dots, x_p) = x_q$ . To see this, we observe that there exist pairwise distinct  $t_0, \dots, t_m \in \mathcal{T}$  such that  $m > p$  and

$$t_0 = 1 \wedge t_1 = x_1 \wedge \dots \wedge t_p = x_p \wedge t_m = 2 \cdot D(x_1, \dots, x_p) \wedge$$

$$\forall i \in \{p+1, \dots, m\} \exists j, k \in \{0, \dots, i-1\} \quad (t_j + t_k = t_i \vee t_i + t_k = t_j \vee t_j \cdot t_k = t_i)$$

For each ring  $\mathbf{K}$  extending  $\mathbb{Z}$  and for each  $x_1, \dots, x_p \in \mathbf{K}$  there exists a unique tuple  $(x_{p+1}, \dots, x_{\text{card}(\mathcal{T})}) \in \mathbf{K}^{\text{card}(\mathcal{T})-p}$  such that the tuple  $(x_1, \dots, x_p, x_{p+1}, \dots, x_{\text{card}(\mathcal{T})})$  solves the system  $\mathcal{H}$ . The sought elements  $x_{p+1}, \dots, x_{\text{card}(\mathcal{T})}$  are given by the formula

$$\forall s \in \{p+1, \dots, \text{card}(\mathcal{T})\} \quad x_s = \tau(s)(x_1, \dots, x_p)$$

**Lemma 1.** *The system  $\mathcal{H} \cup \{x_q + x_q = x_q\}$  can be simply computed. For each ring  $\mathbf{K}$  extending  $\mathbb{Z}$ , the equation  $D(x_1, \dots, x_p) = 0$  is equivalent to the system  $\mathcal{H} \cup \{x_q + x_q = x_q\} \subseteq E_{\text{card}(\mathcal{T})}$ . Formally, this equivalence can be written as*

$$\forall x_1, \dots, x_p \in \mathbf{K} \left( D(x_1, \dots, x_p) = 0 \iff \exists x_{p+1}, \dots, x_{\text{card}(\mathcal{T})} \in \mathbf{K} \right.$$

$$\left. (x_1, \dots, x_p, x_{p+1}, \dots, x_{\text{card}(\mathcal{T})}) \text{ solves the system } \mathcal{H} \cup \{x_q + x_q = x_q\} \right)$$

*For each ring  $\mathbf{K}$  extending  $\mathbb{Z}$  and for each  $x_1, \dots, x_p \in \mathbf{K}$  with  $D(x_1, \dots, x_p) = 0$  there exists a unique tuple  $(x_{p+1}, \dots, x_{\text{card}(\mathcal{T})}) \in \mathbf{K}^{\text{card}(\mathcal{T})-p}$  such that the tuple  $(x_1, \dots, x_p, x_{p+1}, \dots, x_{\text{card}(\mathcal{T})})$  solves the system  $\mathcal{H} \cup \{x_q + x_q = x_q\}$ . Hence, for each ring  $\mathbf{K}$  extending  $\mathbb{Z}$  the equation  $D(x_1, \dots, x_p) = 0$  has the same number of solutions as the system  $\mathcal{H} \cup \{x_q + x_q = x_q\}$ .*

Putting  $M = M/2$  we obtain new families  $\mathcal{T}$  and  $\mathcal{H}$ . There is a unique  $q \in \{1, \dots, \text{card}(\mathcal{T})\}$  such that

$$(q \in \{1, \dots, p\} \wedge x_q = D(x_1, \dots, x_p)) \vee$$

$$(q \in \{p+1, \dots, \text{card}(\mathcal{T})\} \wedge \tau(q) = D(x_1, \dots, x_p))$$

The new system  $\mathcal{H} \cup \{x_q + x_q = x_q\}$  is equivalent to  $D(x_1, \dots, x_p) = 0$  and can be simply computed.

**Theorem 1.** *If Matiyasevich's conjecture is true, then for every computable function  $f : \mathbb{N} \rightarrow \mathbb{N}$  there is a positive integer  $m(f)$  such that for each integer  $n \geq m(f)$  there exists a system  $S \subseteq E_n$  which has at least  $f(n)$  and at most finitely many solutions in integers  $x_1, \dots, x_n$ .*

*Proof.* By Matiyasevich's conjecture, the function  $\mathbb{N} \ni n \rightarrow f(n)! \in \mathbb{N}$  has a finite-fold Diophantine representation. It means that there is a polynomial  $W(x_1, x_2, x_3, \dots, x_r)$  with integer coefficients such that for each non-negative integers  $x_1, x_2$ ,

$$x_1 = f(x_2)! \iff \exists x_3, \dots, x_r \in \mathbb{N} \ W(x_1, x_2, x_3, \dots, x_r) = 0 \quad (\text{E1})$$

and

only finitely many tuples  $(x_3, \dots, x_r) \in \mathbb{N}^{r-2}$  satisfy  $W(x_1, x_2, x_3, \dots, x_r) = 0$  (A).

By the equivalence (E1) and Lagrange's four-square theorem, for each integers  $x_1, x_2$ , the conjunction  $(x_2 \geq 0) \wedge (x_1 = f(x_2)!)$  holds true if and only if there exist integers

$$a, b, c, d, \alpha, \beta, \gamma, \delta, x_3, x_{3,1}, x_{3,2}, x_{3,3}, x_{3,4}, \dots, x_r, x_{r,1}, x_{r,2}, x_{r,3}, x_{r,4}$$

such that

$$W^2(x_1, x_2, x_3, \dots, x_r) + (x_1 - a^2 - b^2 - c^2 - d^2)^2 + (x_2 - \alpha^2 - \beta^2 - \gamma^2 - \delta^2)^2 + (x_3 - x_{3,1}^2 - x_{3,2}^2 - x_{3,3}^2 - x_{3,4}^2)^2 + \dots + (x_r - x_{r,1}^2 - x_{r,2}^2 - x_{r,3}^2 - x_{r,4}^2)^2 = 0$$

The sentence (A) guarantees that for each integers  $x_1, x_2$ , only finitely many integer tuples

$$(a, b, c, d, \alpha, \beta, \gamma, \delta, x_3, x_{3,1}, x_{3,2}, x_{3,3}, x_{3,4}, \dots, x_r, x_{r,1}, x_{r,2}, x_{r,3}, x_{r,4})$$

satisfy the last equality. By Lemma 1, there is an integer  $s \geq 3$  such that for each integers  $x_1, x_2$ ,

$$(x_2 \geq 0 \wedge x_1 = f(x_2)!) \iff \exists x_3, \dots, x_s \in \mathbb{Z} \ \Psi(x_1, x_2, x_3, \dots, x_s) \quad (\text{E2})$$

where the formula  $\Psi(x_1, x_2, x_3, \dots, x_s)$  is algorithmically determined as a conjunction of formulae of the forms  $x_i = 1$ ,  $x_i + x_j = x_k$ ,  $x_i \cdot x_j = x_k$  ( $i, j, k \in \{1, \dots, s\}$ ) and for each integers  $x_1, x_2$  at most finitely many integer tuples  $(x_3, \dots, x_s)$  satisfy  $\Psi(x_1, x_2, x_3, \dots, x_s)$ . Let  $m(f) = 8 + 2s$ , and let  $[\cdot]$  denote the integer part function. For each integer  $n \geq m(f)$ ,

$$n - \left\lfloor \frac{n}{2} \right\rfloor - 4 - s \geq m(f) - \left\lfloor \frac{m(f)}{2} \right\rfloor - 4 - s \geq m(f) - \frac{m(f)}{2} - 4 - s = 0$$

Let  $S$  denote the following system

$$\left\{ \begin{array}{l} \text{all equations occurring in } \Psi(x_1, x_2, x_3, \dots, x_s) \\ n - \left\lfloor \frac{n}{2} \right\rfloor - 4 - s \text{ equations of the form } z_i = 1 \\ \begin{array}{rcl} t_1 & = & 1 \\ t_1 + t_1 & = & t_2 \\ t_2 + t_1 & = & t_3 \\ & \dots & \\ t_{\left\lfloor \frac{n}{2} \right\rfloor - 1} + t_1 & = & t_{\left\lfloor \frac{n}{2} \right\rfloor} \\ t_{\left\lfloor \frac{n}{2} \right\rfloor} + t_{\left\lfloor \frac{n}{2} \right\rfloor} & = & w \\ w + y & = & x_2 \\ y + y & = & y \text{ (if } n \text{ is even)} \\ y & = & 1 \text{ (if } n \text{ is odd)} \\ u \cdot v & = & x_1 \end{array} \end{array} \right.$$

with  $n$  variables. By the equivalence (E2), the system  $S$  is consistent over  $\mathbb{Z}$ . If an integer  $n$ -tuple  $(x_1, x_2, x_3, \dots, x_s, \dots, w, y, u, v)$  solves  $S$ , then by the equivalence (E2),

$$x_1 = f(x_2)! = f(w + y)! = f\left(2 \cdot \left\lfloor \frac{n}{2} \right\rfloor + y\right)! = f(n)!$$

If  $f(n) = 0$ , then the equation  $u \cdot v = x_1 = f(n)! = 1$  has at least  $f(n)$  and at most finitely many solutions in integers  $u, v$ . If  $f(n) \geq 1$  and  $u \in \{1, \dots, f(n)\}$ , then  $u$  divides  $f(n)!$ . Hence, the equation  $u \cdot v = x_1 = f(n)!$  has at least  $f(n)$  and at most finitely many solutions in integers  $u, v$ . In both cases, the conclusion transfers to integer solutions of  $S$ .  $\square$

If we do not assume Matiyasevich's conjecture, then the system  $S$  is still consistent over  $\mathbb{Z}$ , but may have infinitely many integer solutions. Always, if an integer  $n$ -tuple  $(x_1, x_2, x_3, \dots, x_s, \dots, w, y, u, v)$  solves  $S$ , then  $x_1 = f(n)!$ . By choosing a rapidly growing function  $f : \mathbb{N} \rightarrow \mathbb{N}$ , we can guarantee that each integer solution of  $S$  is very large.

### 3. Matiyasevich's conjecture vs the author's conjecture on integer arithmetic

Matiyasevich's conjecture remains in contradiction to the following Conjecture due to the author, see Theorem 2.

**Conjecture** ([2], [11]). *If a system  $S \subseteq E_n$  has only finitely many solutions in integers  $x_1, \dots, x_n$ , then each such solution  $(x_1, \dots, x_n)$  satisfies  $|x_1|, \dots, |x_n| \leq 2^{2^{n-1}}$ .*

**Observation 2.** *For  $n \geq 2$ , the bound  $2^{2^{n-1}}$  cannot be decreased because the system*

$$\left\{ \begin{array}{lcl} x_1 + x_1 & = & x_2 \\ x_1 \cdot x_1 & = & x_2 \\ x_2 \cdot x_2 & = & x_3 \\ x_3 \cdot x_3 & = & x_4 \\ & \dots & \\ x_{n-1} \cdot x_{n-1} & = & x_n \end{array} \right.$$

*has exactly two integer solutions, namely  $(0, \dots, 0)$  and  $(2, 4, 16, 256, \dots, 2^{2^{n-2}}, 2^{2^{n-1}})$ .*

**Theorem 2.** *The Conjecture formulated for an arbitrary computable bound  $\beta : \mathbb{N} \setminus \{0\} \rightarrow \mathbb{N}$  instead of the bound  $\mathbb{N} \setminus \{0\} \ni n \rightarrow 2^{2^{n-1}} \in \mathbb{N}$  remains in contradiction to Matiyasevich's conjecture.*

*Proof.* Assume that the reformulated Conjecture is true. Then, if a system  $S \subseteq E_n$  has only finitely many solutions in integers  $x_1, \dots, x_n$ , then the number of solutions does not exceed  $(1 + 2 \cdot \beta(n))^n$ . Assume that Matiyasevich's conjecture is true. By applying Theorem 1 for  $f(n) = (1 + 2 \cdot \beta(n))^n + 1$ , we conclude that for a sufficiently large value of  $n$ , there is a system  $S \subseteq E_n$  which has at least  $(1 + 2 \cdot \beta(n))^n + 1$  and at most finitely many solutions in integers  $x_1, \dots, x_n$ , a contradiction.  $\square$

#### 4. On the author's conjecture

The Conjecture implies that if equation (2) has only finitely many solutions in integers (non-negative integers, rationals), then their heights are bounded from above by a computable function of

$$\max\left(\{k, n\} \cup \left\{|a(i_1, \dots, i_k)| : (i_1, \dots, i_k \in \{0, \dots, n\}) \wedge (i_1 + \dots + i_k \leq n)\right\}\right)$$

see [11].

To each system  $S \subseteq E_n$  we assign the system  $\widetilde{S}$  defined by

$$(S \setminus \{x_i = 1 : i \in \{1, \dots, n\}\}) \cup \{x_i \cdot x_j = x_j : i, j \in \{1, \dots, n\} \text{ and the equation } x_i = 1 \text{ belongs to } S\}$$

In other words, in order to obtain  $\widetilde{S}$  we remove from  $S$  each equation  $x_i = 1$  and replace it by the following  $n$  equations:

$$\begin{aligned} x_i \cdot x_1 &= x_1 \\ &\dots \\ x_i \cdot x_n &= x_n \end{aligned}$$

**Lemma 2.** *For each system  $S \subseteq E_n$*

$$\begin{aligned} &\{(x_1, \dots, x_n) \in \mathbb{Z}^n : (x_1, \dots, x_n) \text{ solves } \widetilde{S}\} = \\ &\{(x_1, \dots, x_n) \in \mathbb{Z}^n : (x_1, \dots, x_n) \text{ solves } S\} \cup \{(0, \dots, 0)\} \end{aligned}$$

**Corollary.** *The Conjecture is equivalent to  $\forall n \Lambda_n$ , where  $\Lambda_n$  denote the statement*

$$\begin{aligned} & \forall x_1, \dots, x_n \in \mathbb{Z} \exists y_1, \dots, y_n \in \mathbb{Z} \\ & (2^{2^{n-1}} < |x_1| \implies (|x_1| < |y_1| \vee \dots \vee |x_1| < |y_n|)) \wedge \\ & (\forall i, j, k \in \{1, \dots, n\} (x_i + x_j = x_k \implies y_i + y_j = y_k)) \wedge \\ & (\forall i, j, k \in \{1, \dots, n\} (x_i \cdot x_j = x_k \implies y_i \cdot y_j = y_k)) \end{aligned}$$

**Lemma 3.** *For all positive integers  $n, m$  with  $n \leq m$ , if the statement  $\Lambda_n$  fails for  $(x_1, \dots, x_n) \in \mathbb{Z}^n$  and  $2^{2^{m-1}} < |x_1| \leq 2^{2^m}$ , then the statement  $\Lambda_m$  fails for  $(\underbrace{x_1, \dots, x_1}_{m-n+1 \text{ times}}, x_2, \dots, x_n) \in \mathbb{Z}^m$ .*

By the Corollary and Lemma 3, the Conjecture is equivalent to  $\forall n \Psi_n$ , where  $\Psi_n$  denote the statement

$$\begin{aligned} & \forall x_1, \dots, x_n \in \mathbb{Z} \exists y_1, \dots, y_n \in \mathbb{Z} \\ & (2^{2^{n-1}} < |x_1| = \max(|x_1|, \dots, |x_n|) \leq 2^{2^n} \implies (|x_1| < |y_1| \vee \dots \vee |x_1| < |y_n|)) \wedge \\ & (\forall i, j, k \in \{1, \dots, n\} (x_i + x_j = x_k \implies y_i + y_j = y_k)) \wedge \\ & (\forall i, j, k \in \{1, \dots, n\} (x_i \cdot x_j = x_k \implies y_i \cdot y_j = y_k)) \end{aligned}$$

In contradistinction to the statements  $\Lambda_n$ , each true statement  $\Psi_n$  can be confirmed by a brute-force search in a finite amount of time.

Let  $T_n$  denote the set of all integer tuples  $(a_1, \dots, a_n)$  for which there exists a system  $S \subseteq E_n$  such that  $(a_1, \dots, a_n)$  solves  $S$  and  $S$  has at most finitely many solutions in integers  $x_1, \dots, x_n$ . If  $(a_1, \dots, a_n) \in T_n$ , then  $(a_1, \dots, a_n)$  solves the system

$$\begin{cases} x_i = 1 & (\text{all } i \in \{1, \dots, n\} \text{ with } a_i = 1) \\ x_i + x_j = x_k & (\text{all } i, j, k \in \{1, \dots, n\} \text{ with } a_i + a_j = a_k) \\ x_i \cdot x_j = x_k & (\text{all } i, j, k \in \{1, \dots, n\} \text{ with } a_i \cdot a_j = a_k) \end{cases}$$

which has only finitely many solutions in integers  $x_1, \dots, x_n$ .



**Theorem 3.** *The Conjecture is true for  $n \leq 3$ .*

*Proof.*  $T_1 = \{0, 1\}$ .  $T_2$  consists of the pairs  $(0, 0), (1, 1), (-1, 1), (0, 1), (1, 2), (2, 4)$  and their permutations.  $T_3$  consists of the triples

$$\begin{aligned} & (0, 0, 0), (1, 1, 1), \\ & (-1, -1, 1), (0, 0, 1), (1, 1, -1), (1, 1, 0), (1, 1, 2), (2, 2, 1), (2, 2, 4), (4, 4, 2), \\ & (1, -2, -1), (1, -1, 0), (1, -1, 2), (1, 0, 2), (1, 2, 3), (1, 2, 4), \\ & (2, 4, -2), (2, 4, 0), (2, 4, 6), (2, 4, 8), (2, 4, 16), \\ & (-4, -2, 2), (-2, -1, 2), (3, 6, 9), (4, 8, 16) \end{aligned}$$

and their permutations. □

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